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2003 J. Phys. A: Math. Gen. 36 7715

(http://iopscience.iop.org/0305-4470/36/28/303)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 7715–7731 PII: S0305-4470(03)58947-8

The automorphisms of Novikov algebras in low dimensions

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Received 28 January 2003, in final form 16 May 2003
Published 1 July 2003
Online at stacks.iop.org/JPhysA/36/7715

Abstract

Novikov algebras were introduced in connection with Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus. They also correspond to a class of vertex algebras. An automorphism of a Novikov algebra is a linear isomorphism φ satisfying $\varphi(xy) = \varphi(x)\varphi(y)$ which keeps the algebraic structure. The set of automorphisms of a Novikov algebra is a Lie group whose Lie algebra is just the Novikov algebra's derivation algebra. The theory of automorphisms plays an important role in the study of Novikov algebras. In this paper, we study the automorphisms of Novikov algebras. We get some results on their properties and classification in low dimensions. These results are fundamental in a certain sense, and they will serve as a guide for further development. Moreover, we apply these results to classify Gel'fand–Dorfman bialgebras and Novikov–Poisson algebras. These results also can be used to study certain phase spaces and geometric classical r-matrices.

PACS numbers: 02.20.Sv, 02.20.−a, 02.30.−f, 02.40.−k

1. Introduction

A Novikov algebra A is a vector space over a field $\mathbf{F}$ with a bilinear product $(x, y) \to xy$ satisfying

$$(x, y, z) = (y, x, z) \tag{1.1}$$

and

$$(xy)z = (xz)y \tag{1.2}$$

for $x, y, z \in A$, where

$$(x, y, z) = (xy)z - x(yz). \tag{1.3}$$

Novikov algebras were introduced firstly as a kind of algebraic systems related to a class of operators in the formal calculus of variations to have the Hamiltonian property [1–3]. The study of Hamiltonian operators plays an important role in the integrability of certain nonlinear partial differential equations. Following [3], the differential operator H with matrix components H_{ij}:

$$H_{ij} = \sum_{k=1}^{N} \left(c_{ij}^k u_k^{(1)} + \left(c_{ij}^k + c_{ji}^k \right) u_k^{(0)} \frac{\mathrm{d}}{\mathrm{d}x} \right) \tag{1.4}$$

is Hamiltonian if and only if $\left\{c_{ij}^k\right\}$ is the set of structure constants of a Novikov algebra, that is, let $\{e_1, \ldots, e_N\}$ be a basis of the algebra, and $e_i e_j = \sum_{k=1}^N c_{ij}^k e_k$. From another point of view, Novikov algebras were also introduced in connection with the following Poisson brackets of hydrodynamic type [4–6]:

$$\{u(x), v(y)\} = \partial_x((uv)(x))\delta(x-y) + (uv + vu)\partial_x \delta(x-y). \tag{1.5}$$

Furthermore, let A be a Novikov algebra and set

$$\hat{A} = A \times \mathbf{F}[t, t^{-1}] \tag{1.6}$$

where t is an indeterminate. Define a bracket operation on $\hat{A}$ by

$$[u \otimes t^m, v \otimes t^n] = muv \otimes t^{m+n-1} - nvu \otimes t^{m+n-1} \qquad \forall u, v \in A \quad m, n \in \mathbf{Z}. \tag{1.7}$$

Then $(\hat{A}, [,])$ forms a Lie algebra. The Lie algebras constructed from Novikov algebras as above can induce a class of vertex Lie algebras and vertex algebras, which are fundamental algebraic structures in conformal field theory [7–10]. Moreover, the vertex algebras satisfying certain conditions must correspond to some Novikov algebras (roughly speaking, such a vertex algebra V is generated from $V_{(2)}$ = a Novikov algebra as vector spaces, with some additional conditions) [11].

In fact, the name 'Novikov algebra' was given by Osborn [12, 13]. Moreover, on the other hand, Novikov algebras are a special class of left-symmetric algebras which only satisfy equation (1.1). Left-symmetric algebras are non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [14–17].

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y] = xy - yx \tag{1.8}$$

defines a (sub-adjacent) Lie algebra $\mathcal{G} = \mathcal{G}(A)$. It is isomorphic to a subalgebra of Lie algebra $\hat{A}$ by letting $m = n = 1$ in equation (1.7). Let L_x, R_x denote the left and right multiplication operators, respectively, i.e., $L_x(y) = xy, R_x(y) = yx, \forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative. If every R_x is nilpotent, then A is called right-nilpotent or transitive. The transitivity corresponds to the completeness of the affine manifolds in geometry [14, 15].

There has been some progress in the study of Novikov algebras, such as the fundamental structure theory of a finite-dimensional Novikov algebra over an algebraically closed field with characteristic 0 [18], the infinite-dimensional simple Novikov algebras [12, 13], the finite-dimensional simple Novikov algebras over an algebraically closed field with prime characteristic [19], the Poisson structures on Novikov algebras [20], the classification of Novikov algebras over the complex number field in low dimensions [21], the realization of Novikov algebras [22, 23], the invariant bilinear forms on Novikov algebras [24, 25], the fermionic Novikov algebras [26] and so on. However, due to the non-associativity, there are still many open questions which are quite different from any known algebras.

Among them, one of the most important topics is automorphism which plays an important role not only in algebra itself, but also in many related fields. Obviously, its first important use is to classify Novikov algebras in the sense of algebraic isomorphisms. Furthermore, the set of automorphisms of a Novikov algebra is a Lie group which will be useful to relate the study of Novikov algebras from pure algebra to geometry, which will lead to some applications in physics.

In this paper, we study the automorphisms of Novikov algebras. The paper is organized as follows. In section 2, we briefly give some basic properties of automorphisms of Novikov algebras. We also discuss how to obtain some interesting automorphisms on a lot of Novikov algebras based on a kind of realization theory of Novikov algebras. In section 3, we give the classification of automorphisms and inner automorphisms of three-dimensional Novikov algebras over **C**. In section 4, we discuss the classification of some bialgebras such as Gel'fand–Dorfman bialgebras and Novikov–Poisson algebras. In section 5, we apply the classification results in sections 2 and 3 to study certain phase spaces and geometric classical *r*-matrices. In section 6, we give some conclusions based on the discussion in the previous sections.

Throughout the paper, the algebras that we consider are of finite dimension.

2. The automorphism group of a Novikov algebra

Let A be a Novikov algebra. Let φ be a linear isomorphism of A, then φ is an automorphism of A if and only if

$$\varphi(xy) = \varphi(x)\varphi(y) \qquad \forall x, y \in A. \tag{2.1}$$

It is well known [27] that the set $\mathrm{Aut}(A)$ of automorphisms of A is a Lie group over the real field **R** or the complex field **C** under the product

$$(\varphi_1\varphi_2)(x) = \varphi_1(\varphi_2(x)) \qquad \forall x \in A \qquad \varphi_1, \varphi_2 \in \mathrm{Aut}(A). \tag{2.2}$$

Let $\mathrm{End}(A)$ denote the set of all linear transformations of A. Then $\mathrm{End}(A)$ is a Lie algebra with respect to the bracket

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_1\mathcal{A}_2 - \mathcal{A}_2\mathcal{A}_1 \qquad \forall \mathcal{A}_1, \mathcal{A}_2 \in \mathrm{End}(A). \tag{2.3}$$

A derivation D of A is a linear transformation $D \in \mathrm{End}(A)$ satisfying

$$D(xy) = (Dx)y + x(Dy) \qquad \forall x, y \in A. \tag{2.4}$$

It is also well known that the set $\mathrm{D}(A)$ of all derivations of A is a Lie subalgebra of $\mathrm{End}(A)$. Moreover, $\mathrm{D}(A)$ is the Lie algebra of the automorphism group $\mathrm{Aut}(A)$ [27] since $\exp tD$ is an automorphism of A for any $D \in \mathrm{D}(A)$, and $t \in \mathbf{R}$ or $t \in \mathbf{C}$.

In fact, the above discussion holds for any non-associative algebra. The automorphisms of Novikov algebras have many common properties which belong to all non-associative algebras. Moreover, it is easy to see that any automorphism of a Novikov algebra is also an automorphism of its sub-adjacent Lie algebra.

In general, the automorphism group $\mathrm{Aut}(A)$ is very complicated and it is also difficult to obtain non-trivial automorphisms. However, based on a kind of realization theory of Novikov algebras in [22, 23], we can obtain some interesting automorphisms on a lot of Novikov algebras. Next we discuss them in detail. To make this work self-contained, we give a brief introduction at first.

Let A be a commutative associative algebra with the product $(,\cdot,)$ and D be a derivation. Then the new product

$$x *_a y = x \cdot Dy + a \cdot x \cdot y \tag{2.5}$$

makes $(A, *_a)$ a Novikov algebra for $a = 0$ by Gel'fand [3], for $a \in \mathbf{F}$ by Filipov [28] and for a fixed element $a \in A$ by Xu [20]. In [22], we show that the algebra $(A, *) = (A, *_0)$ given by Gel'fand is transitive, and the other two kinds of Novikov algebras given by Filipov and Xu are special deformations of the former. Moreover, in [22, 23] a deformation theory of Novikov algebras is constructed and we prove that the Novikov algebras in dimension $\leqslant 3$ can be realized as the algebras defined by Gel'fand and their compatible infinitesimal deformations. We conjecture that this conclusion is still true in higher dimensions. In particular, in dimensions 2 and 3, many transitive Novikov algebras (except (A6) with $l = 0$, (A8), (A10)) and almost every non-transitive Novikov algebra (except only (E1)) can be realized through equation (2.5).

Proposition 1. *Let φ be an automorphism of $(A, \cdot)$. If $\varphi D = D\varphi$, then φ is an automorphism of $(A, *_a)$ for $a = 0$ or $a \in \mathbf{F}$. In particular, in this case, $\exp tD \in \mathrm{Aut}(A, *_a)$ for every $t \in \mathbf{F}$. For $a \in A$, if $\varphi D = D\varphi$ and $\varphi(a) = a$, then φ is an automorphism of $(A, *_a)$.*

Proof. For $a = 0$ or $a \in \mathbf{F}$, we have

$$\begin{aligned}\varphi(x *_a y) &= \varphi(x \cdot Dy + a \cdot x \cdot y)\\ &= \varphi(x) \cdot \varphi(Dy) + a \cdot \varphi(x) \cdot \varphi(y)\\ &= \varphi(x) *_a \varphi(y) + \varphi(x) \cdot (\varphi D - D\varphi)(y).\end{aligned}$$

Similarly, for $a \in A$, we have

$$\varphi(x *_a y) = \varphi(x) *_{\varphi(a)} \varphi(y) + \varphi(x) \cdot (\varphi D - D\varphi)(y).$$

Hence, the proposition holds. □

Corollary 1. *The centralizer*

$$C_{GL(A)}(\exp tD) = \{\varphi \in GL(A) | \varphi \exp tD = \exp tD\varphi\} \subset \mathrm{Aut}(A, *_a) \tag{2.6}$$

for $a = 0$ or $a \in \mathbf{F}$, where $GL(A)$ is the set of all invertible linear transformations on A. For $a \in A$, the automorphism group of $(A, *_a)$ contains the intersection of the centralizer $C_{GL(A)}(\exp tD)$ and the isotropic subgroup of $GL(A)$ at a, that is,

$$C_{GL(A)}(\exp tD) \cap \{\varphi \in GL(A) | \varphi(a) = a\} \subset \mathrm{Aut}(A, *_a). \tag{2.7}$$

Let us give the automorphisms of two-dimensional Novikov algebras over the complex number field whose classification is given in [21]. Recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^n c_{11}^k e_k & \cdots & \sum_{k=1}^n c_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^n c_{n1}^k e_k & \cdots & \sum_{k=1}^n c_{nn}^k e_k \end{pmatrix} \tag{2.8}$$

where $\{e_i\}$ is a basis of A and $e_i e_j = \sum_{k=1}^n c_{ij}^k e_k$. Moreover, under the same basis, any automorphism φ of A can be determined by a matrix, that is,

$$\varphi = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad \varphi(e_i) = \sum_{j=1}^{n} a_{ij} e_j \qquad \det \varphi \neq 0. \tag{2.9}$$

For any two-dimenisonal non-commutative Novikov algebra, we have known that it can be realized through equation (2.5) [22, 23]. Thus, we have the following proposition.

Proposition 2. *The automorphisms of two-dimensional Novikov algebras over* **C** *are given as follows:*

Characteristic matrix	Automorphism group	Associated $(A, \cdot)$ and D	Remark
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$GL(2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $a_{11}a_{22} - a_{12}a_{21} \neq 0$		Commutative
(T2) $\begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}$	$\begin{pmatrix} a_{11}^2 & 0 \\ a_{21} & a_{11} \end{pmatrix}, a_{11} \neq 0$		Commutative
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		Commutative
(N2) $\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}, a_{11} \neq 0$		Commutative
(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}, a_{11} \neq 0$		Commutative
(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}, a_{11} \neq 0$	(N3) with $D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = 0$	Aut(T3) = Aut(N3)
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 1 \end{pmatrix}, a_{11} \neq 0$	(N3) with $D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = e_2$	
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ a_{21} & 1 \end{pmatrix}$	(N3) with $D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = e_1 + e_2$	
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}, a_{11} \neq 0$	(N3) with $D = \begin{pmatrix} l-1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = e_2$	Aut(N6) = Aut(N3)

At the end of this section, we discuss the inner automorphisms. The set of inner automorphisms is the most important subset of the automorphism group of a Novikov algebra. A general theory on inner automorphisms is given in [29].

Let A be a Novikov algebra. The Lie-subalgebra $\mathcal{L}(A)$ generated by all linear transformations $L_x, R_y\ (\forall x, y \in A)$ is called the Lie multiplication algebra (or Lie transformation algebra). It is easy to show that the Lie transformation algebra of a Novikov algebra A is

$$\mathcal{L}(A) = L(A) + \mathbf{F}\big[R_{e_1}, R_{e_2}, \ldots, R_{e_n}\big] \tag{2.10}$$

where $e_1, \ldots, e_n$ is a basis of A, $L(A)$ is the set of left multiplications and $\mathbf{F}\big[R_{e_1}, R_{e_2}, \ldots, R_{e_n}\big]$ is the polynomial algebra generated by $R_{e_1}, \ldots, R_{e_n}$.

A derivation D of A is called an inner derivation if $D \in \mathcal{L}(A)$. It is obvious that the set $\mathrm{Inn}(A)$ of all inner derivations is a (Lie) ideal of the Lie algebra $D(A)$. An automorphism

φ of A is inner if φ is contained in the subgroup of $\mathrm{Aut}(A)$ generated by $\exp(\mathrm{Inn}(A))$. Thus $\varphi = \exp D_1 \cdots \exp D_n$, where $D_i \in \mathrm{Inn}(A)$. Let $\mathrm{Int}(A)$ denote the subgroup of inner automorphisms of A. It is a connected Lie group. Through direct computation, we have the following proposition.

Proposition 3. *The inner automorphisms of two-dimensional Novikov algebras are given as follows:*

$$\mathrm{Int(T1)} = \mathrm{Int(N1)} = \mathrm{Int(N2)} = \mathrm{Int(N3)} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \qquad \mathrm{Int(T2)} = \left\{\begin{pmatrix} 1 & 0 \\ a_{21} & 1 \end{pmatrix}\right\}$$

$$\mathrm{Int(T3)} = \mathrm{Int(N6)} = \left\{\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}, a_{11} \neq 0\right\} \qquad \mathrm{Int(N4)} = \left\{\begin{pmatrix} a_{11} & 0 \\ a_{21} & 1 \end{pmatrix}, a_{11} \neq 0\right\}$$

$$\mathrm{Int(N5)} = \left\{\begin{pmatrix} 1 & 0 \\ a_{21} & 1 \end{pmatrix}\right\}.$$

3. The automorphisms and inner automorphisms of three-dimensional Novikov algebras

In this section, we give the automorphism groups and the inner automorphism groups of three-dimensional Novikov algebras over the complex number field **C** for which the classification is given in [21]. We give our main results in the following table:

Characteristic matrix	Automorphism group	Inner automorphism group
(A1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\varphi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \det\varphi \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(A2) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix}$	$\begin{pmatrix} a_{33}^2 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, a_{33}a_{22} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$
(A3) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}$	$\begin{pmatrix} 2a_{22}^2 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & -a_{23} & a_{22} \end{pmatrix};$ $\begin{pmatrix} 2a_{22}^2 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & -a_{22} \end{pmatrix},$ $a_{22}\left(a_{22}^2 + a_{23}^2\right) \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$
(A4) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{33}^3 & 0 & 0 \\ 2a_{32}a_{33} & a_{33}^2 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, a_{33} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$
(A5) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$ $a_{22}a_{33} - a_{23}a_{32} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$
(A6) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & le_1 \end{pmatrix}$	$\begin{pmatrix} a_{22}^2 + la_{23}^2 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & -la_{23} & a_{22} \end{pmatrix},$ $a_{22}^2 + la_{23}^2 \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$

Characteristic matrix	Automorphism group	Inner automorphism group
(A7) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & le_1 & e_2 \end{pmatrix}$ $l \neq 1$	$\begin{pmatrix} a_{33}^3 & 0 & 0 \\ (l+1)a_{32}a_{33} & a_{33}^2 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, a_{33} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ (l+1)a_{32} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}$
(A8) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{33}^3 & 0 & 0 \\ a_{32}a_{33} & a_{33}^2 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, a_{33} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ a_{32} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}$
(A9) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{11}a_{22} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{22} \neq 0$
(A10) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{22} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{22} \neq 0$
(A11) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 0 \end{pmatrix}$ $\lvert l \rvert \leqslant 1, l \neq 0$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} l = 1,$ $a_{11}a_{22} - a_{12}a_{21} \neq 0;$ $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} l \neq 1,$ $a_{11}a_{22} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix} l = 1, a_{11} \neq 0$ $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11}^l & 0 \\ 0 & 0 & 1 \end{pmatrix} l \neq 1, a_{11} \neq 0$
(A12) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} e^{a_{11}} & 0 & 0 \\ a_{11}e^{a_{11}} & e^{a_{11}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(A13) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{22}^2 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{22} \neq 0$	$\begin{pmatrix} a_{22}^2 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{22} \neq 0$
$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(B1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $a_{11} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(B3) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{11} \neq 0$
(B4) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_1 + e_3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$

Characteristic matrix	Automorphism group	Inner automorphism group
(B5) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$
(C1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$ $a_{11}a_{22} - a_{12}a_{22} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(C2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11}a_{22} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(C3) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{11}a_{22} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{11} \neq 0$
(C4) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 + e_3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{22} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$
(C5) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11}a_{22} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$
(C6) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 1 \end{pmatrix}, a_{11}a_{22} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 1 \end{pmatrix}, a_{22} \neq 0$
(C7) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 + e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & 1 \end{pmatrix}$
(C8) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix},$ $a_{11}a_{22} - a_{12}a_{21} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}, a_{11} \neq 0$
(C9) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 1, 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 1 \end{pmatrix}, a_{11}a_{22} \neq 0$	$\begin{pmatrix} a_{22}^{1-l} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 1 \end{pmatrix}, a_{22} \neq 0$
(C10) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & 0 & e_3 + e_2 \end{pmatrix}$ $l \neq 1$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} l = 0,$ $a_{11} \neq 0$ $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & 1 \end{pmatrix} l \neq 0,$ $a_{11} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} l = 0$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & 1 \end{pmatrix} l \neq 0$
(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$ $a_{11}a_{22} - a_{12}a_{21} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(C12) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & le_2 & e_3 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11}a_{22} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Characteristic matrix	Automorphism group	Inner automorphism group
(C13) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & ke_2 & e_3 \end{pmatrix}$ $l, k \neq 1, 0$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} l = k,$ $a_{11}a_{22} - a_{12}a_{21} \neq 0$ $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} l \neq k,$ $a_{11}a_{22} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix} l = k, a_{11} \neq 0$ $\begin{pmatrix} a_{11}^{l-1} & 0 & 0 \\ 0 & a_{11}^{k-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} l \neq k, a_{11} \neq 0$
(C14) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_1 + e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(C15) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & e_1 + le_2 & e_3 \end{pmatrix}$ $l \neq 1, 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} \mathrm{e}^{a_{21}(l-1)} & 0 & 0 \\ a_{21}\,\mathrm{e}^{a_{21}(l-1)} & \mathrm{e}^{a_{21}(l-1)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(C16) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ a_{31} & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} \mathrm{e}^{a_{11}} & 0 & 0 \\ -a_{11}\,\mathrm{e}^{a_{11}} & \mathrm{e}^{a_{11}} & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$
(C17) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3 + e_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{21} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 1 \end{pmatrix}$
(C18) $\begin{pmatrix} 0 & 0 & e_1 + e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ a_{31} & a_{31} & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{31} & 1 \end{pmatrix}$
(C19) $\begin{pmatrix} 0 & 0 & e_1 + e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3 + e_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{31} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{31} & 1 \end{pmatrix}$
(D1) $\begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(D3) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 + e_2 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(D4) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{1}{2}e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11}^2 & 0 \\ 0 & a_{32} & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11}^2 & 0 \\ 0 & a_{32} & 1 \end{pmatrix}, a_{11} \neq 0$
(D5) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{1}{2}e_1 & 0 & e_3 + e_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & 1 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & 1 \end{pmatrix}$
(D6) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & (2l-1)e_2 & e_3 \end{pmatrix}$ $l \neq \frac{1}{2}, 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$

Characteristic matrix	Automorphism group	Inner automorphism group
(E1) $\begin{pmatrix} 0 & 0 & 0 \\ -e_1 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{11} \neq 0$

4. Classification of Gel'fand–Dorfman bialgebras and Novikov–Poisson algebras

In this section, we discuss the classification of some bialgebraic structures one of which is a Novikov algebra through its automorphism group. The first example is the Gel'fand–Dorfman bialgebra. A Gel'fand–Dorfman bialgebra A is a vector space with two operations '$[\cdot,\cdot], *$' such that $(A, [\cdot,\cdot])$ forms a Lie algebra and $(A, *)$ forms a Novikov algebra (which satisfies equations (1.1) and (1.2)) with the following compatible identity:

$$[x * y, z] - [x * z, y] + [x, y] * z - [x, z] * y - x * [y, z] = 0 \qquad \forall x, y, z \in A. \tag{4.1}$$

This bialgebraic structure was introduced by Gel'fand and Dorfman in studying certain Hamiltonian pairs [3]. In fact, similar to equations (1.4) and (1.5), the corresponding Hamiltonian operator and Poisson bracket are:

$$H_{ij} = \sum_{k=1}^{N}\left(d_{ij}^k u_k^{(0)} + c_{ij}^k u_k^{(1)} + \left(c_{ij}^k + c_{ji}^k\right)u_k^{(0)}\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{4.2}$$

$$\{u(x), v(y)\} = [u, v](x)\delta(x-y) + \partial_x((u*v)(x))\delta(x-y) + (u*v + v*u)\partial_x\delta(x-y) \tag{4.3}$$

respectively, where $\{e_1, \ldots, e_N\}$ is a basis of A and $[e_i, e_j] = \sum_{k=1}^N d_{ij}^k e_k$, $e_i * e_j = \sum_{k=1}^N c_{ij}^k e_k$. Moroever, from a Gel'fand–Dorfman bialgebra $(A, [\cdot,\cdot], *)$, similar to equation (1.7), we also can get a Lie algebra structure on $\hat{A}$ by

$$[u \otimes t^m, v \otimes t^n] = [u, v] \otimes t^{m+n} + mu * v \otimes t^{m+n-1} - nv * u \otimes t^{m+n-1} \qquad \forall u, v \in A \qquad m, n \in \mathbf{Z}. \tag{4.4}$$

It also plays an important role in the study of vertex algebras [30, 31].

Since there are two operations on a Gel'fand–Dorfman bialgebra, it is not easy to obtain the classification in the sense of isomorphisms. Obviously, two Gel'fand–Dorfman bialgebras $(A_i, [\cdot,\cdot], *)$, $i = 1, 2$, are isomorphic if and only if there exists a linear isomorphism $f : A_1 \to A_2$, such that

$$f([a, b]) = [f(a), f(b)] \qquad f(a * b) = f(a) * f(b) \qquad \forall a, b \in A_1. \tag{4.5}$$

Thus we often need to discuss two algebraic isomorphisms simultaneously. Moreover, it is also obvious that a linear transformation of a Gel'fand–Dorfman bialgebra itself is an (algebraic) isomorphism (called an automorphism) if and only if it is an automorphism of both the Novikov algebra and the Lie algebra, that is, it is at the intersection of the automorphism groups of the Novikov algebra and the Lie algebra.

Usually, we need to fix an algebraic system which has been classified at first and then we classify the other algebraic structure which is compatible with the former. In general, we need the following three steps:

Step 1. Classify one algebraic system with structure constants.
Step 2. For the fixed algebraic system, find the compatible structure constants of the second algebraic system.

Step 3. Classify those compatible structure constants of the second algebraic system. Here, we would like to point out that the corresponding linear transformations describing the isomorphic relations among the second algebras with the different structure constants must be in the automorphism group of the first algebraic system.

For a Gel'fand–Dorfman bialgebra, because the structure of the Lie algebra is much simpler than that of the Novikov algebra, we can give the classification of Gel'fand–Dorfman bialgebras as the classification of the compatible Lie algebras for the fixed Novikov algebras.

Let $\{e_i\}$ be a basis of a Gel'fand–Dorfman bialgebra $(A, *, [\cdot,\cdot])$. Then $(A, *, [\cdot,\cdot])$ is determined by the (form) characteristic matrix given as

$$
\begin{aligned}
(\mathcal{A}, *) &= \begin{pmatrix} \sum_{k=1}^{n} c_{11}^k e_k & \cdots & \sum_{k=1}^{n} c_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} c_{n1}^k e_k & \cdots & \sum_{k=1}^{n} c_{nn}^k e_k \end{pmatrix} \\
(\mathcal{A}, [\cdot,\cdot]) &= \begin{pmatrix} \sum_{k=1}^{n} d_{11}^k e_k & \cdots & \sum_{k=1}^{n} d_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} d_{n1}^k e_k & \cdots & \sum_{k=1}^{n} d_{nn}^k e_k \end{pmatrix}
\end{aligned}
\tag{4.6}
$$

where $e_i * e_j = \sum_{k=1}^{n} c_{ij}^k e_k$ and $[e_i, e_j] = \sum_{k=1}^{n} d_{ij}^k e_k$.

For a fixed $(\mathcal{A}, *)$, the elements in $(\mathcal{A}, [\cdot,\cdot])$ should satisfy the following equations:

$$d_{ii}^p = 0 \qquad d_{ij}^p = -d_{ji}^p \qquad \sum_{l=1}^{n} \left(d_{ij}^l d_{lk}^p + d_{jk}^l d_{li}^p + d_{ki}^l d_{lj}^p \right) = 0 \qquad p = 1, \ldots, n \tag{4.7}$$

$$\sum_{l=1}^{n} \left(d_{ij}^l c_{lk}^p - d_{ik}^l c_{lj}^p + d_{lk}^p c_{ij}^l - d_{lj}^p c_{ik}^l - d_{jk}^l c_{il}^p \right) = 0 \qquad p = 1, \ldots, n. \tag{4.8}$$

From the automorphism groups of Novikov algebras given in section 2 and through equations (4.7) and (4.8) and direct computation, we have the following proposition.

Proposition 4. *The classification of Gel'fand–Dorfman bialgebras in dimension 2 is given in the following table (for clear expressions, we indicate the results of the three steps of classification in three columns, respectively):*

Characteristic matrix $(\mathcal{A}, *)$	Compatible characteristic matrix $(\mathcal{A}, [\cdot,\cdot])$	Characteristic matrix $(\mathcal{A}, [\cdot,\cdot])$
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 + d_{12}^2 e_2 \\ -d_{12}^1 e_1 - d_{12}^2 e_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix}$
(T2) $\begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix}$
(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
(N2) $\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Characteristic matrix $(\mathcal{A}, *)$	Compatible characteristic matrix $(\mathcal{A}, [\cdot,\cdot])$	Characteristic matrix $(\mathcal{A}, [\cdot,\cdot])$
(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & d_{12}^1 e_1 \\ -d_{12}^1 e_1 & 0 \end{pmatrix}$

On the other hand, there is another bi-algebraic structure, which is a commutative associative algebra structure compatible with a Novikov algebra structure. It is the so-called Novikov–Poisson algebra. A Novikov–Poisson algebra A is a vector space with two operations '$\cdot, \circ$' such that $(A, \cdot)$ forms a commutative associative algebra (which may not have an identity element) and $(A, \circ)$ forms a Novikov algebra with the compatible identities

$$(x \cdot y) \circ z = x \cdot (y \circ z) = y \cdot (x \circ z) \tag{4.9}$$

$$(x \circ y) \cdot z - x \circ (y \cdot z) = (y \circ x) \cdot z - y \circ (x \cdot z) \tag{4.10}$$

for $x, y, z \in A$. Novikov–Poisson algebras were introduced to construct a tensor theory since in general the tensor product of two arbitrary Novikov algebras is not a Novikov algebra. But the tensor product of two Novikov–Poisson algebras is still a Novikov–Poisson algebra. Moreover, there exists a Hamiltonian superoperator associated with a Novikov–Poisson algebra $(A, \cdot, \circ)$ with an identity element 1 in $(A, \cdot)$ such that $1 \circ 1 = 2$ [19, 20].

As in the case of Gel'fand–Dorfman bialgebras, we can also give the classification of Novikov–Poisson algebras as the classification of the compatible commutative associative algebras for the fixed Novikov algebras. In fact, let $(A, \cdot, \circ)$ be a Novikov–Poisson algebra with a basis $\{e_i\}$, and $e_i \circ e_j = \sum_{k=1}^n c_{ij}^k e_k$, $e_i \cdot e_j = \sum_{k=1}^n d_{ij}^k e_k$. The corresponding equations which are parallel to equations (4.7) and (4.8) are

$$d_{ij}^p = d_{ji}^p, \sum_{l=1}^n d_{ij}^l d_{lk}^p = \sum_{l=1}^n d_{jk}^l d_{il}^p \qquad p = 1, \ldots, n \tag{4.11}$$

$$\sum_{l=1}^n d_{ij}^l c_{lk}^p = \sum_{l=1}^n c_{jk}^l d_{il}^p \qquad p = 1, \ldots, n \tag{4.12}$$

$$\sum_{l=1}^n \left(c_{ij}^l d_{lk}^p - d_{jk}^l c_{il}^p\right) = \sum_{l=1}^n \left(c_{ji}^l d_{lk}^p - d_{ik}^l c_{jl}^p\right) \qquad p = 1, \ldots, n. \tag{4.13}$$

Proposition 5. *The classification of Novikov–Poisson algebras in dimension 2 is given in the following table* ($m, n \in \mathbf{C}$)*:*

Characteristic matrix $(\mathcal{A}, \circ)$	Compatible characteristic matrix $(\mathcal{A}, \cdot)$	Characteristic matrix $(\mathcal{A}, \cdot)$
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} d_{11}^1e_1 + d_{11}^2e_1 & d_{12}^1e_1 + d_{12}^2e_2 \\ d_{12}^1e_1 + d_{12}^2e_2 & d_{22}^1e_1 + d_{22}^2e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}, \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$
(T2) $\begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}$	$\begin{pmatrix} 0 & me_2 \\ me_2 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & me_1 \end{pmatrix}, \begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$
(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & me_1 \\ me_1 & e_1 + me_2 \end{pmatrix}$
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} ne_1 & 0 \\ 0 & me_2 \end{pmatrix}$	$\begin{pmatrix} ne_1 & 0 \\ 0 & me_2 \end{pmatrix}, m \geqslant n$
(N2) $\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} ne_1 & 0 \\ 0 & me_2 \end{pmatrix}$	$\begin{pmatrix} e_1 & 0 \\ 0 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & me_2 \end{pmatrix}$
(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_2 \\ me_2 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & me_1 \\ me_1 & e_1 + me_2 \end{pmatrix}$
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}$
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & me_1 \end{pmatrix}$
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} 0 & me_1 \\ me_1 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & me_1 \\ me_1 & e_1 + me_2 \end{pmatrix}$

5. The equivalences of certain phase spaces and geometric classical *r*-matrices

In the section we will see that the isomorphisms of left-symmetric algebras can induce the equivalence of their corresponding phase spaces and geometric classical r-matrices. Hence the results in sections 2 and 3 on Novikov algebras (as a special class of left-symmetric algebras) can be applied to get the classification of equivalent maps of certain phase spaces and geometric classical r-matrices.

Let $\mathcal{G}$ be a Lie algebra. According to [32, 33], a phase space $T^*\mathcal{G}$ satisfies the following conditions: (1) $T^*\mathcal{G} = \mathcal{G} \oplus \mathcal{G}^*$ as the direct sum of vector spaces, where $\mathcal{G}^*$ is the dual space of $\mathcal{G}$; (2) $T^*\mathcal{G}$ is a Lie algebra such that the symplectic form ω defined by

$$\omega(u + u^*, v + v^*) = u^*(v) - v^*(u) \qquad \forall u, v \in \mathcal{G} \qquad u^*, v^* \in \mathcal{G}^* \tag{5.1}$$

is a 2-cocycle on $T^*\mathcal{G}$, that is,

$$\omega([u_1 + u_1^*, u_2 + u_2^*], u_3 + u_3^*) + \mathrm{CP} = 0 \qquad u_i \in \mathcal{G}, u_i^* \in \mathcal{G}^* \tag{5.2}$$

where 'CP' stands for 'cyclic permutation'.

On the other hand, for a Lie algebra $\mathcal{G}$, let $r \in \mathcal{G} \otimes \mathcal{G}$. Then r is a solution of the classical Yang–Baxter equation on $\mathcal{G}$ if and only if r satisfies [34–36]

$$[r_{12}, r_{13}] + [r_{12}, r_{13}] + [r_{13}, r_{23}] = 0 \qquad \text{in } U(\mathcal{G}) \tag{5.3}$$

where $U(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$ and for $r = \sum_i a_i \otimes b_i$,

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1 \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \tag{5.4}$$

Moreover, r is called skew-symmetric if

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i). \tag{5.5}$$

Let X be a smooth, affine algebraic variety over $\mathbf{C}$. In [37, 38], a geometric classical r-matrix is defined as a derivation $r : \mathbf{C}[X \times X] \to \mathbf{C}[X \times X]$, which satisfies the classical Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{13}] + [r_{13}, r_{23}] = 0 \qquad \text{in } \mathbf{C}[X \times X \times X] \tag{5.6}$$

and the unitarity condition

$$r + r_{21} = 0 \qquad \text{in } \mathbf{C}[X \times X]. \tag{5.7}$$

In fact, from [37], it is easy to know that a geometric classical r-matrix corresponds to a skew-symmetric solution of the classical Yang–Baxter equation on $\mathcal{G} \ltimes V^*$ which is in $\mathcal{G} \otimes V^* - V^* \otimes \mathcal{G}$, where $\mathcal{G}$ is a Lie algebra, $\rho : \mathcal{G} \to gl(V)$ is a representation of $\mathcal{G}$ with $\dim V = \dim \mathcal{G}$ and $\rho^* : \mathcal{G} \to gl(V^*)$ is its dual representation. Moreover, $\mathcal{G} \ltimes V^*$ is a Lie algebra given by

$$[x + u^*, y + v^*] = [x, y] + \rho^*(x)v^* - \rho^*(y)u^* \qquad \forall x, y \in \mathcal{G} \qquad u^*, v^* \in V^*. \tag{5.8}$$

The relation between phase spaces and geometric classical r-matrices can be given through left-symmetric algebras as follows:

Proposition 6. *Let $\mathcal{G}$ be a Lie algebra. Then the following conditions are equivalent:*

(1) There is a left-symmetric algebra structure on $\mathcal{G}$;

(2) There is a bijective 1-cocycle for $\mathcal{G}$. That is, there is a representation $\rho : \mathcal{G} \to gl(V)$ satisfying $\dim V = \dim \mathcal{G}$ *and a linear isomorphism q from $\mathcal{G}$ onto V such that*

$$q([x, y]) = \rho(x)q(y) - \rho(y)q(x) \qquad \forall x, y \in \mathcal{G}. \tag{5.9}$$

(3) $T^*\mathcal{G}$ is a phase space such that the Lie bracket on $T^*\mathcal{G}$ is given by $\mathcal{G} \ltimes \mathcal{G}^*$, that is,

$$[u_1 + u_1^*, u_2 + u_2^*] = [u_1, u_2] + \rho^*(u_1)u_2^* - \rho^*(u_2)u_1^* \qquad \forall u_1, u_2 \in \mathcal{G} \qquad u_1^*, u_2^* \in \mathcal{G}^* \tag{5.10}$$

where $\rho : \mathcal{G} \to gl(\mathcal{G})$ is a representation of $\mathcal{G}$ and $\rho^* : \mathcal{G} \to gl(\mathcal{G}^*)$ is its dual representation.

(4) There is a geometric classical r-matrix. That is, there is a Lie algebra $\mathcal{G}'$ with $\dim \mathcal{G}' = \dim \mathcal{G}$ *and $\mathcal{G}$ is a representation space of $\mathcal{G}'$ such that there exists a skew-symmetric solution of classical Yang–Baxter equation on $\mathcal{G}' \ltimes \mathcal{G}^*$ which is in $\mathcal{G}' \otimes \mathcal{G}^* - \mathcal{G}^* \otimes \mathcal{G}'$ and the Lie algebra structure on $\mathcal{G}' \ltimes \mathcal{G}^*$ is given by equation (5.8).*

Proof. In fact, we can get this proposition from the following references: '(1)$\Leftrightarrow$ (2)' has been proved in [14]; '(1)$\Leftrightarrow$ (3)' has been proved in [32]; '(2)$\Leftrightarrow$ (4)' has been proved in [37]. However, in order to give a more explicit picture, we briefly repeat these results through their relations with left-symmetric algebras:

'(1) $\Rightarrow$ (2)' $L : \mathcal{G} \to gl(\mathcal{G})$ given by $L(x) = L_x$ is a representation and id is a 1-cocycle.

'(2) $\Rightarrow$ (1)' The left-symmetric algebra structure on $\mathcal{G}$ is defined by

$$xy = q^{-1}(\rho(x)q(y)) \qquad \forall x, y \in \mathcal{G}. \tag{5.11}$$

'(1) $\Rightarrow$ (3)' Let $\rho = L$, it is easy to know that the symplectic form ω defined by equation (5.1) is a 2-cocycle. Hence $T^*\mathcal{G}$ is a phase space.

'(3) $\Rightarrow$ (1)' Since the symplectic form ω is a 2-cocycle on $T^*\mathcal{G}$, we can get

$$[u_1, u_2] = \rho(u_1)u_2 - \rho(u_2)u_1 \qquad \forall u_1, u_2 \in \mathcal{G}. \tag{5.12}$$

Hence $u_1 u_2 = \rho(u_1)u_2$ defines a left-symmetric algebra structure on $\mathcal{G}$.
'(1) $\Rightarrow$ (4)' Let $\mathcal{G}' = \mathcal{G}$. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathcal{G}$ and $\{e_1^*, \ldots, e_n^*\}$ be its dual basis, that is, $e_i^*(e_j) = \delta_{ij}$. Then it is easy to show that

$$r = \sum_{i=1}^{n}(e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{5.13}$$

satisfies the classical Yang–Baxter equation, where the representation of $\mathcal{G}$ is given by $L : \mathcal{G} \to gl(\mathcal{G})$.
'(4) $\Rightarrow$ (1)' Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathcal{G}$ and $\{e_1^*, \ldots, e_n^*\}$ be its dual basis. Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathcal{G}'$. Then we can set

$$r = \sum_{i,j=1}^{n}(a_{ij}x_i \otimes e_j^* - a_{ij}e_j^* \otimes x_i). \tag{5.14}$$

There is a left-symmetric algebra structure on $\mathcal{G}$ defined by

$$e_i e_j = \sum_{k=1}^{n} a_{ki}\rho(x_k)e_j. \tag{5.15}$$

□

Let A be a Novikov algebra, then it is a left-symmetric algebra satisfying an additional condition $R_x R_y = R_y R_x$. Hence from the above proposition, we have

Corollary 2. *Let A be a Novikov algebra. Then we have*

(1) The corresponding bijective 1-cocycle satisfies an additional condition:

$$\rho(q^{-1}(\rho(x)q(y))q(z) = \rho(q^{-1}(\rho(x)q(z))q(y) \qquad \forall x, y, z \in A. \tag{5.16}$$

(2) The corresponding phase space T^*A with the representation ρ satisfies an additional condition:

$$\rho(\rho(u_1)u_2)u_3 = \rho(\rho(u_1)u_3)u_2 \qquad \forall u_1, u_2, u_3 \in A. \tag{5.17}$$

(3) The corresponding geometric classical r-matrix satisfies an additional condition

$$\sum_{l,m,s=1}^{n}\left(a_{li}a_{sm}b_{lj}^{m}b_{sk}^{t} - a_{li}a_{sm}b_{lk}^{m}b_{sj}^{t}\right) = 0 \qquad t = 1, \ldots, n, \tag{5.18}$$

where $r = \sum_{i,j=1}^{n}(a_{ij}x_i \otimes e_j^* - a_{ij}e_j^* \otimes x_i)$ and $\rho(x_i)e_j = \sum_k b_{ij}^k e_k$.

On the other hand, it is natural to define that two bijective 1-cocycles (or phase spaces, or geometric classical r-matrices) are equivalent if and only if their corresponding left-symmetric algebras are isomorphic. For example, two bijective 1-cocycles $(\mathcal{G}_i, V_i, q_i)$ are equivalent if and only if there exists a linear isomorphism $F : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$q_2 F q_1^{-1}\rho_1(x)q_1(y) = \rho_2(F(x))q_2F(y) \qquad \forall x, y \in \mathcal{G}_1. \tag{5.19}$$

Corollary 3. *The classification of automorphisms of Novikov algebras in dimensions 2 and 3 gives the classification of equivalent maps of their corresponding phase spaces and geometric classical r-matrices.*

6. Discussion and conclusions

From the results given in the previous sections, we can obtain:

1. In dimensions 2 and 3, the dimension of automorphism group of any Novikov algebra which is not semisimple (the direct sum of fields) is >0. It should be true in higher dimensions.
2. In dimensions 2 and 3, the inner automorphism group of a non-commutative algebra is non-trivial, in particular, the inner automorphism group of a non-associative Novikov algebra is non-trivial. And $\dim \mathrm{Int}(A) \leqslant \dim A$. Furthermore, when A is a Novikov algebra in dimension 2 or 3 and A is neither type (N4) nor type (C8), we have $\dim \mathrm{Int}(A) < \dim A$.
3. In dimensions 2 and 3, $\mathrm{Aut}(A) = \mathrm{Int}(A) \neq \{1\}$ if and only if A is isomorphic to one of the following types:

   (T3), (N4), (N5), (N6), (A10), (A13), (B3), (B4), (B5), (C19), (D3), (D4), (D6), (E1).

   We would like to point out that there exists a non-inner automorphism on the Novikov algebra of type (D5), although all of its derivations are inner.
4. Obviously the automorphisms of two Novikov–Poisson algebra can induce an automorphism of their tensor product algebra which is still a Novikov–Poisson algebra [20]. However, in general, it is not true that every automorphism of the tensor product algebra is induced as above.

At the end of this paper, we also would like to point out that it is interesting to see that the dimensions of automorphism groups of certain infinite-dimensional Novikov algebras are finite [13].

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China, Mathematics Tianyuan Foundation, K C Wong Education Foundation. We thank Professor S P Novikov and Professor I M Gel'fand for useful suggestions and great encouragement and Professor P Etingof, Professor B A Kupershmidt and Professor X Xu for communicating their research in this field and the generous help. We also express our sincere gratitude to the referees for carefully reading the original manuscript which had contained a few gaps. The first author (CB) also thanks Professor J Lepowsky, Professor Y Z Huang and Professor H S Li for the hospitality extended to him during his stay at Rutgers, The State University of New Jersey and for valuable discussions.

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